Triangle inequality

Mason Kamb asked a question about the triangle inequality. I think this is a formulation of his conjecture.

Theorem 1. *Let V be a real vector space (not necessisarily finite dimensional). Suppose $f : V \longrightarrow \mathbb{R}$ satisfies:*

1. $f(v) > 0 \iff v \neq 0$.
2. $f(tv) = |t|f(v),\ t \in \mathbb{R}$.
3. $\{v : f(v) < 1\}$ *is convex.*

Then f satisfies the triangle inquality:

$$f(u+v) \leq f(u) + f(v).$$

Proof. Let $B = \{v : f(v) < 1\}$. By rescaling we can assume $u, v \in B$. Let $\epsilon > 0$ be given. Then there are numbers s, t that make the following inequalities valid:

$$0 < f(u) < s < f(u) + \epsilon < 1, \tag{1}$$
$$0 < f(v) < t < f(v) + \epsilon < 1. \tag{2}$$

Let

$$\tilde{u} = \frac{u}{s},\ \tilde{v} = \frac{v}{t}.$$

Then $\tilde{u}, \tilde{v}, s\tilde{u}, t\tilde{v} \in B$. By convexity

$$\frac{s\tilde{u} + t\tilde{v}}{s+t} \in B.$$

Hence $u + v = s\tilde{u} + t\tilde{v} \in (s+t)B$. So

$$f(u+v) < s + t < f(u) + f(v) + 2\epsilon.$$

Since ϵ is arbitrary the theorem is true. □

We can use f to define a norm on V.